JOURNAL OF APPROXIMATION THEORY 9, 1–19 (1973)

Optimal Starting Approximations for Iterative Schemes

G. MEINARDUS

Institut für Angewandte Mathematik der Universität Erlangen, West Germany, and Michigan State University, East Lansing, Michigan 48823

AND

G. D. TAYLOR*

Michigan State University, East Lansing, Michigan 48823

1. INTRODUCTION

Let X be a compact subset of the closed interval $[a, b]$ and let $C(X)$ denote the space of all continuous real-valued functions defined on X, normed by

$$\|f\| = \max\{|w(x) f(x)| : x \in X\}.$$

Here $w(x)$ is a fixed element of $C(x)$, positive throughout X. If $w \equiv 1$ then we shall write $\|\cdot\|_\infty$. Let K be a convex subset of $C(X)$ and Φ a continuous mapping of K into $C(X)$. Fixing a subset M of K, we shall be interested in the problem of approximating $g \in \Phi(K)$ with elements from $\Phi(M)$. With some additional assumptions on Φ, g, K, and M we shall be able to develop a theory for this nonlinear approximation problem which is quite similar to the standard Chebyshev theory. Because of the applications of this theory to iterative processes that can be used to compute the value of a function (such as $x^{1/N}$) we shall use the following terminology: if $p \in M$ has the property that

$$\|g - \Phi(p)\| \leqslant \|g - \Phi(q)\|$$

for all $q \in M$ then we shall say that p is a best starting approximation for g (with respect to Φ and M).

A specific example of this theory is the following. Let $x = [a, b]$, $a > 0$, $K = \{f \in C[a, b]: f > 0\}$, $w(x) = x^{-1/2}$ and $\Phi(h)(x) = (1/2)(h(x) + x/h(x))$, where Φ is a single Newton iteration for calculating $x^{1/2}$ starting with $h(x)$.

* Supported in part by NSF Grant GP-12088.

Set $M = \pi_n \cap K$ where π_n consists of all polynomials of degree less than or equal to n (some fixed n). Then for $g(x) = x^{1/2}$ the above approximation problem reduces to that of relative approximation of $x^{1/2}$ with functions $(1/2)(p(x) + x/p(x))$. This problem was first studied by Moursund [12], who showed that there exists a unique $p^* \in M$ minimizing

$$\left\| \frac{x^{1/2} - \Phi(p)(x)}{x^{1/2}} \right\|_\infty$$

over all $p \in M$. Also, he showed that p^* is the unique element of M minimizing

$$\left\| \frac{x^{1/2} - \Phi^m(p)(x)}{x^{1/2}} \right\|_\infty$$

over all $p \in M$, where $\Phi^k(h) \equiv \Phi(\Phi^{k-1}(h))$, $\Phi^1 = \Phi$. More recently, Sterbenz and Fike [14] and King and Phillips [6] showed (independently) that p^* is a multiple of $\tilde{p}$, the best relative approximation to $x^{1/2}$ on $[a, b]$. Finally, a somewhat more numerical study of this problem has been done by Holzwarth [4].

2. General Theory

In this section, we shall show that if Φ satisfies certain conditions then the behavior encountered in the above special problem is preserved in the more general one.

DEFINITION 1. The operator Φ is said to be pointwise strictly monotone at $f \in K$ provided for each $h, k \in K$ we have

$$|\Phi(h)(x_0) - \Phi(f)(x_0)| < |\Phi(k)(x_0) - \Phi(f)(x_0)|$$

for every $x_0 \in X$ at either $k(x_0) < h(x_0) \leqslant f(x_0)$ or $f(x_0) \leqslant h(x_0) < k(x_0)$.

LEMMA 1. *Let $\Phi: K \to C(X)$ be pointwise strictly monotone at $f \in K$. If $k \in K$ and at $x_0 \in X$, $k(x_0) \neq f(x_0)$ then $\Phi(k)(x_0) \neq \Phi(f)(x_0)$.*

Proof. In the above definition let $h \equiv f$. Then $|\Phi(k)(x_0) - \Phi(f)(x_0)| > 0$.

DEFINITION 2. The operator Φ is said to be pointwise fixed on $f \in K$ provided $h \in K$, $x_0 \in X$ and $h(x_0) = f(x_0)$ imply $\Phi(h)(x_0) = \Phi(f)(x_0)$.

Let us give an example of a pointwise strictly monotone operator at f which is not pointwise fixed on f. Define $K \subset C[0, 1]$ by

$$K = \{f(x) = ax^2 : 0 \leqslant a \leqslant 1\}$$

and $\Phi: K \to C[0, 1]$ by $\Phi(ax^2) = x^2 + (1 - a)$. Now Φ is continuous and pointwise strictly monotone at x^2. Indeed, suppose $ax_0^2 < bx_0^2 \leqslant x_0^2$; then $x_0 \neq 0$ and $a < b \leqslant 1$ so that $| \Phi(ax^2)(x_0) - x_0^2 | = 1 - a > 1 - b = | \Phi(bx^2)(x_0) - x_0^2 |$ as $\Phi(x^2) = x^2$. Each $ax^2 \in K$ has the property that $ax^2 = x^2$ at $x = 0$; but $\Phi(ax)(0) \neq 0$ if $a \neq 1$ so that Φ is not pointwise fixed on x^2.

Next, we wish to show that the composition of two continuous operators whose domains and ranges match up correctly and both of which possess the above two properties is again such an operator.

LEMMA 2. *Let $\Phi: K \to C(X)$ and $\Psi: L \to C(X)$ be continuous operators. Suppose that $\Phi(K) \subset L$, Φ is pointwise strictly monotone and pointwise fixed at $f \in K$ and Ψ is pointwise strictly monotone and pointwise fixed at $\Phi(f) \in L$. Then $\Psi\Phi: K \to C(X)$ is a continuous pointwise strictly monotone operator at f which is also pointwise fixed on f.*

Proof. The only property of $\Psi\Phi$ that needs to be shown is the pointwise strict monotonicity at f, as the other properties follow trivally. We first prove that if $h, k \in K$ satisfy $k(x_0) < h(x_0) \leqslant f(x_0)$ or $f(x_0) \leqslant h(x_0) < k(x_0)$ then either $\Phi(k)(x_0) < \Phi(h)(x_0) \leqslant g(x_0)$ or $g(x_0) \leqslant \Phi(h)(x_0) < \Phi(k)(x_0)$ where $\Phi(f) = g$. From the pointwise strict monotonicity of Φ we have that $| \Phi(k)(x_0) - g(x_0)| > | \Phi(h)(x_0) - g(x_0)|$. Suppose that $\Phi(k)(x_0) < g(x_0) < \Phi(h)(x_0)$ or $\Phi(h)(x_0) < g(x_0) < \Phi(k)(x_0)$. Let $l_\alpha(x) = \alpha h(x) + (1 - \alpha)\, k(x)$, $0 \leqslant \alpha \leqslant 1$. Then $l_\alpha \in K$ and $\Phi(l_\alpha)(x_0)$ is a continuous function of α. Thus, by the intermediate value theorem we know that there exists an $\alpha_0 \in (0, 1)$ such that $\Phi(l_{\alpha_0})(x_0) = g(x_0)$. This implies $l_{\alpha_0}(x_0) = f(x_0)$ which is a contradiction. Thus, we must have either $\Phi(k)(x_0) < \Phi(h)(x_0) \leqslant g(x_0)$ or $g(x_0) \leqslant \Phi(h)(x_0) < \Phi(k)(x_0)$. From this it follows immediately that $| \Psi\Phi(k)(x_0) - \Psi\Phi(f)(x_0)| > | \Psi\Phi(h)(x_0) - \Psi\Phi(f)(x_0)|$.

One special case should be mentioned here, namely, the case where $\Phi: K \to K$. For example, Newton's iteration for calculating the value of a nice function may be viewed as such an operator. This will be discussed in a later section. One very nice property of such operators is stated in the next Lemma.

LEMMA 3. *If $\Phi: K \to K$ is a continuous operator on the convex set K which is pointwise strictly monotone at $f \in K$ then Φ is pointwise fixed on f.*

Proof. Suppose $h \in K$ and, for some $x_0 \in X$, $h(x_0) = f(x_0)$. Now if every function $k \in K$ has the property that $k(x_0) = f(x_0)$ then, necessarily $\Phi(h)(x_0) = \Phi(f)(x_0)$. Assume there exists $k \in K$ such that $k(x_0) \neq f(x_0)$. Let $k_\alpha = (1 - \alpha)k + \alpha f$. Then $k_\alpha(x_0) \neq f(x_0)$ for $\alpha \in (0, 1)$. On the other hand, $k_\alpha \to f$ as $\alpha \to 1$, so that $\Phi(k_\alpha) \to \Phi(f)$. Since norm convergence implies pointwise convergence we must have $\Phi(k_\alpha)(x_0) \to \Phi(f)(x_0)$. The assumption that Φ is pointwise strictly monotone at f implies that

$$|\Phi(k_\alpha)(x_0) - \Phi(f)(x_0)| > |\Phi(h)(x_0) - \Phi(f)(x_0)| \qquad \text{for all} \quad \alpha \in (0, 1)$$

so that $\Phi(h)(x_0) = \Phi(f)(x_0)$.

Because of the connection of this theory to iterative processes used on a computer for evaluating special functions, we wish to state the following corollary to Lemma 2.

COROLLARY 1. *If Φ: $K \to K$ is continuous, $\Phi(f) = f$ for some $f \in K$ and Φ is pointwise strictly monotone at f then Φ^m: $K \to K$ ($\Phi^m(h) = \Phi(\Phi^{m-1}(h))$, $m = 2, 3,\ldots$, $\Phi^1 = \Phi$) is pointwise strictly monotone at f and $\Phi^m(f) = f$.*

In closing this section we wish to state an existence theorem which for specific operators is sometimes difficult to check, but for the general case is a standard statement.

THEOREM 1. *Let K be a convex subset of $C(X)$, $M \subset K$ and let Φ: $K \to C(X)$ be continuous. Then corresponding to each $g \in \Phi(K)$ there exists $p^* \in M$ minimizing $\| g - \Phi(p)\|$ over all $p \in M$, provided there exists a compact subset M_1 of M and a positive constant η such that $q \in M \sim M_1$ implies* $\inf\{\| g - \Phi(p)\| : p \in M\} + \eta \leqslant \| g - \Phi(q)\|$.

Theorem 1 follows simply from the fact that a continuous real function on a compact set assumes there its minimum. However, as we shall see in the examples to be studied later, the actual application of this theorem is somewhat tedious.

3. CHARACTERIZATION THEOREMS

In this section we shall study the problem of characterizing best starting approximations for special choices of K and M. We shall establish an alternation type theory for each case considered and from this theory conclude that the best starting approximation (if it exists) is unique. In what follows, we assume that $X \subset [a, b]$ has at least $n + 1$ points.

THEOREM 2. *Let Φ: $K \to C(X)$ be a continuous operator. Let V be an n-dimensional Haar subspace of $C[a, b]$ and let $M = K \cap V$ be a nonempty relatively open subset of V. Finally assume that Φ is pointwise strictly monotone and pointwise fixed at $f \in K \sim M$. Then $p \in M$ is a best starting approximation* *for* $\Phi(f)$ *if and only if there exist a sequence* $\{x_i\}_1^{n+1} \subset X$ *for which* $x_1 < x_2 < \cdots < x_{n+1}$,

$$| w(x_i)(\Phi(f)(x_i) - \Phi(p)(x_i))| = \| \Phi(f) - \Phi(p)\|$$

and

$$\operatorname{sgn}(f(x_i) - p(x_i)) = (-1)^{i+1} \operatorname{sgn}(f(x_1) - p(x_1)).$$

Proof. Since $f \notin M$, there exists a point $x_0 \in X$ for which $f(x_0) \neq p(x_0)$. Thus $\| \Phi(f) - \Phi(p)\| \neq 0$. Also, for some choices of Φ, the family $\Phi(M)$ is not unisolvent so that this alternation theorem does not follow from the general theory of unisolvent families. Now suppose that

$$| w(x_i)(\Phi(f)(x_i) - \Phi(p)(x_i))| = \| \Phi(f) - \Phi(p)\|$$

and

$$\operatorname{sgn}(f(x_i) - p(x_i)) = (-1)^{i+1} \operatorname{sgn}(f(x_1) - p(x_1)), \qquad i = 1,..., n + 1.$$

Then $\Phi(f)(x_i) \neq \Phi(p)(x_i)$. Let $q \in M$ be such that $\| \Phi(f) - \Phi(q)\| \leqslant \| \Phi(f) - \Phi(p)\|$. Fix an i, $1 \leqslant i \leqslant n + 1$. Then $| w(x_i)(\Phi(f)(x_i) - \Phi(q)(x_i))| \leqslant | w(x_i)(\Phi(f)(x_i) - \Phi(q)(x_i))|$. Now, either $p(x_i) > f(x_i)$ or $p(x_i) < f(x_i)$. In the first case we must have $p(x_i) \geqslant q(x_i)$ and in the second $p(x_i) \leqslant q(x_i)$ by the pointwise strict monotonicity of Φ at f. But this implies that $p \equiv q$ as in the standard theory.

Now suppose that p is a best starting approximation for $\Phi(f)$ and that the desired behavior is exhibited on a sequence $\{x_i\}_1^k \subset X$ with $x_1 < x_2 < \cdots < x_k$ where k is maximal and $k \leqslant n$. First of all let us dispose of the case where the error curve is constant. That is, $w(x)(\Phi(f)(x) - \Phi(p)(x)) = \| \Phi(f) - \Phi(p)\|$ for all $x \in X$ (or $-\| \Phi(f) - \Phi(p)\|$) and $\operatorname{sgn}(f(x) - p(x))$ is constant. Assume the first occurs. In this case $p(x) \neq f(x)$ for any $x \in X$ as $\Phi(f)(x) \neq \Phi(p)(x)$ for each $x \in X$. Since V is an n-dimensional Haar subspace of $C[a, b]$ we know that there exists $q \in V$ such that $q > 0$ on $[a, b]$. Therefore, there exists a real λ such that $p + \lambda q$ is strictly between f and p. Also, since M is relatively open in V we have that $p + \lambda q \in M$ for sufficiently small $| \lambda |$. By the pointwise strict monotonicity of Φ at f and the continuity of all functions involved we have that

$$\| \Phi(f) - \Phi(p)\| > \| \Phi(f) - \Phi(p + \lambda q)\|,$$

which implies that a best starting approximation cannot have a constant error curve.

Thus, we may assume that there exists $\{x_i\}_1^k \subset X$, $k \leqslant n$ on which

$$| w(x_i)(\Phi(f)(x_i) - \Phi(p)(x_i)) = \| \Phi(f) - \Phi(p)\|$$

and

$$\operatorname{sgn}(f(x_i) - p(x_i)) = (-1)^{i+1} \operatorname{sgn}(f(x_1) - p(x_1)).$$

Now subdivide $[a, b]$ into relatively open intervals $I_1, \ldots, I_k$ such that $x_i \in I_i$, the set of extreme points, i.e.,

$$\{x \in X: | w(x)(\Phi(f)(x) - \Phi(p)(x))| = \| \Phi(f) - \Phi(p)\|\}$$

is a subset of $\bigcup_{i=1}^{k} I_i$, $\bar{I}_j \cap \bar{I}_r = \varnothing$ for $j \neq r$, and at all extreme points in each I_i the function $f - p$ has a constant sign. Let

$$Y = X \cap \left(\bigcup_{i=1}^{k} \tilde{I}_i\right).$$

Y is a compact subset of X and $| w(x)(\Phi(f)(x) - \Phi(p)(x))| < \| \Phi(f) - \Phi(p)\|$ for all $x \in Y$. Thus, by continuity there exists $\rho > 0$ for which

$$\max_{x \in Y} | w(x)(\Phi(f)(x) - \Phi(p)(x))| \leqslant \| \Phi(f) - \Phi(p)\| - \rho.$$

Next, let

$$W_i = \{x \in X \cap \bar{I}_i: | w(x)(\Phi(f)(x) - \Phi(p)(x))| \geqslant \| \Phi(f) - \Phi(p)\|/2$$
$$\text{and } \operatorname{sgn}(f(x) - p(x)) = \operatorname{sgn}(f(x_i) - p(x_i))\}.$$

Then $W = \bigcup_{i=1}^{k} W_i$ is a compact subset of X and by continuity there exists an $\eta > 0$ such that $|f(x) - p(x)| \geqslant \eta$ on W. Let

$$Z_i = \{x \in X \cap \bar{I}_i: | w(x)(\Phi(f)(x) - (p)(x))| \geqslant \| \Phi(f) - \Phi(p)\|/2$$
$$\text{and } \operatorname{sgn}(f(x) - p(x)) \neq \operatorname{sgn}(f(x_i) - p(x_i))\}$$

and let $Z = \bigcup_{i=1}^{k} Z_i$. Note that $| w(x)(\Phi(f)(x) - \Phi(p)(x))| < \| \Phi(f) - \Phi(p)\|$ for all $x \in Z$ by the definition of the intervals I_i. Finally, let

$$U_i = \{x \in X \cap \bar{I}_i: | w(x)(\Phi(f)(x) - \Phi(p)(x))| \leqslant \| \Phi(f) - \Phi(p)\|/2\}$$

and let $U = \bigcup_{i=1}^{k} U_i$. Then, by continuity there exists $\delta > 0$, $\delta \leqslant \rho$ such that

$$\max_{x \in Z \cup U} | w(x)(\Phi(f)(x) - \Phi(p)(x))| \leqslant \| \Phi(f) - \Phi(p)\| - \delta.$$

Now by the standard Haar theory, there exists $q \in V$ such that

$$\operatorname{sgn} q(x) = \operatorname{sgn}(f(x_i) - p(x_i)) \qquad \text{for all} \quad x \in I_i, \qquad i = 1, \ldots, k.$$

By the continuity of Φ we can select a $\lambda_1 > 0$ such that $| \lambda | \leqslant \lambda_1$ implies

$$\max_{x \in Y \cup Z \cup U} | w(x)(\Phi(f)(x) - \Phi(p + \lambda q)(x))| \leqslant \| \Phi(f) - \Phi(p)\| - \delta/2.$$

Next, by continuity of f and p we can select a λ_2, $0 < \lambda_2 \leqslant \lambda_1$, such that $0 < \lambda < \lambda_2$ implies that $p + \lambda q$ is strictly between f and p on $W = \bigcup_{i=1}^{k} W_i$. Thus, by the strict monotonicity of Φ at f we have

$$\max_{x \in W} | w(x)(\Phi(f)(x) - \Phi(p + \lambda q)(x))| < \| \Phi(f) - \Phi(p)\|.$$

Combining all these results, we conclude that there exists a $\lambda > 0$ for which

$$\| \Phi(f) - \Phi(p + \lambda q)\| < \| \Phi(f) - \Phi(p)\|.$$

Finally, for sufficiently small $\lambda > 0$, $p + \lambda q \in M$ since M is open in V. Thus $p + \lambda q$ is a better starting approximation that p and we have arrived at a contradiction. This completes the proof of the Theorem.

COROLLARY 2. *If $\Phi(f)$ has a starting approximation under the setting of Theorem 2 then it is unique.*

For our next example we shall define K as follows:

$$K = \{f \in C(X) \colon l(x) \leqslant f(x) \leqslant u(x)\}$$

where $l, u \in C(X)$ satisfy $l < u$. Let V be an n-dimensional Haar subspace of $C[a, b]$ and set $M = K \cap V$ which we assume is nonempty.

THEOREM 3. *Let $\Phi\colon K \to C(X)$ be a continuous operator which is pointwise strictly monotone and pointwise fixed at $f \in K \sim M$. Then $p \in M$ is a best starting approximation for $\Phi(f)$ if and only if there exists $\{x_i\}_1^{n+1}$ for which*

(a) $x_1 < x_2 < \cdots < x_{n+1}$,

(b) $| w(x_i)(\Phi(f)(x_i) - \Phi(p)(x_i))| = \| \Phi(f) - \Phi(p)\|$, $p(x_i) = u(x_i)$ *or* $p(x_i) = l(x_i)$,

(c) $\operatorname{sgn}^*(f(x_i) - p(x_i)) = (-1)^{i+1} \operatorname{sgn}^*(f(x_1) - p(x_1))$

where

$$\operatorname{sgn}^*(f(x) - p(x)) = \begin{cases} \operatorname{sgn}(f(x) - p(x)) & \text{if } p(x) \neq l(x) \text{ and } p(x) \neq u(x), \\ +1 & \text{if } p(x) = l(x) \\ -1 & \text{if } p(x) = u(x). \end{cases}$$

Proof. Suppose $p \in M$ has the above properties on the set $\{x_i\}_1^{n+1}$. Let $q \in M$ be such that $\| \Phi(f) - \Phi(q)\| \leqslant \| \Phi(f) - \Phi(p)\|$. If $p(x_i) = l(x_i)$ then $p(x_i) \leqslant q(x_i)$ since $q \in M$. If $p(x_i) = u(x_i)$ then $p(x_i) \geqslant q(x_i)$. If

$$| w(x_i)(\Phi(f)(x_i) - \Phi(p)(x_i))| = \| \Phi(f) - \Phi(p)\|$$

then $|w(x_i)(\Phi(f)(x_i) - \Phi(p)(x_i))| \geq |w(x_i)(\Phi(f)(x_i) - \Phi(q)(x_i))|$. Since $f \notin M$ we have that $\|\Phi(f) - \Phi(p)\| > 0$ so that $\text{sgn}^*(f(x_i) - p(x_i)) \neq 0$. Suppose $\text{sgn}^*(f(x_i) - p(x_i)) = +1$ then $f(x_i) > p(x_i)$ and by the pointwise strict monotonicity of Φ at f we must have $q(x_i) \geqslant p(x_i)$. Likewise, if $\text{sgn}^*(f(x_i) - p(x_i)) = -1$ then $q(x_i) \leqslant p(x_i)$. Thus

$$\text{sgn}^*(f(x_1) - p(x_1))(-1)^i(p(x_i) - q(x_i)) \geqslant 0, \qquad i = 1,..., n+1.$$

This implies $p \equiv q$ so that p is the best starting approximation for $\Phi(f)$.

Now suppose that $p \in M$ is a best starting approximation for $\Phi(f)$ and that p has the desired behavior on a set of points $\{x_i\}_1^k \subset X$, $x_1 < x_2 < \cdots < x_k$ where k is a maximal and $k \leqslant n$. Let us first show that it is not possible that $\Phi(f)(x) - \Phi(p)(x) \equiv \lambda$, $|\lambda| > 0$ for all $x \in X$. For if this were the case then we would have that $p \neq f$ for each $x \in X$ so that $p + \lambda q$ is strictly between p and f for sufficiently small λ of proper sign and $q \in V$ is such that $q > 0$ on $[a, b]$. Thus, we have that $p + \lambda q \in M$ and $\|\Phi(f) - \Phi(p + \lambda q)\| < \|\Phi(f) - \Phi(p)\|$ which is a contradiction.

Let $I_1, ..., I_k$ be a collection of open intervals in $[a, b]$ such that $x_i \in I_i$, $\bar{I}_i \cap \bar{I}_j = \varnothing$, for $i \neq j$, all extreme points $= \{x \in X: |w(x)(\Phi(f)(x) - \Phi(p)(x))| = \|\Phi(f) - \Phi(p)\|,\ p(x) = l(x) \text{ or } p(x) = u(x)\}$ are contained in $\bigcup_1^k I_i$ and for each extreme point in I_i the function $\text{sgn}^*(f(x) - p(x))$ has the same value. Now select $q \in V$ such that $\text{sgn}\, q(x) = \text{sgn}^*(f(x_i) - p(x_i))$ for all $x \in I_i$, $i = 1,..., k$. This we can do since $k \leqslant n$. Consider $p + \lambda q$ for real λ. We shall show that there exists λ_0 such that $p + \lambda_0 q \in M$ and

$$\|\Phi(f) - \Phi(p + \lambda_0 q)\| < \|\Phi(f) - \Phi(p)\|.$$

Let $Y = X \cap (\bigcup_{i=1}^k \bar{I}_i)$; Y is a compact subset of X and

$$|w(x)(\Phi(f)(x) - \Phi(p)(x))| < \|\Phi(f) - \Phi(p)\| \qquad \text{for all} \quad x \in Y.$$

Thus, by continuity there exists $\lambda_1 > 0$ for which $0 < \lambda < \lambda_1$ implies

$$\max_{x \in Y} |w(x)(\Phi(f)(x) - \Phi(p + \lambda q)(x))| < \|\Phi(f) - \Phi(p)\|.$$

Also, $p(x)$ differs from both $l(x)$ and $u(x)$ on Y so that there exists λ_2 such that $0 < \lambda_2 \leqslant \lambda_1$ and $0 < \lambda \leqslant \lambda_2$ implies

$$l(x) \leqslant p(x) + \lambda q(x) \leqslant u(x), \qquad x \in Y.$$

Next, let

$$W_i = \{x \in X \cap \bar{I}_i: |w(x)(\Phi(f)(x) - \Phi(p)(x))| \geqslant \|\Phi(f) - \Phi(p)\|/2 \\ \text{and } \text{sgn}(f(x) - p(x)) = \text{sgn}^*(f(x_i) - p(x_i))\}$$

and

$$V_i = \{x \in X \cap \bar{I}_i: p(x) = l(x) \text{ or } p(x) = u(x)\}.$$

Then all extreme points contained in I_i are contained in $W_i \cup V_i$ and by the definition of q, the strict monotonicity of Φ at f and continuity there exists a λ_3 satisfying $0 < \lambda_3 \leqslant \lambda_2$, for which $x \in W_i \cup V_i$ and $0 < \lambda \leqslant \lambda_3$ imply

$$\max_{x \in W_i \cup V_i} | w(x)(\Phi(f)(x) - \Phi(p + \lambda q)(x))| < \| \Phi(f) - \Phi(p)\|$$

and

$$l(x) \leqslant p(x) + \lambda q(x) \leqslant u(x).$$

Do this for each $i = 1,..., k$ and let λ_4 be a positive real number for which the above holds for all i. Let

$$\begin{aligned} Z_i = \{x \in X \cap \bar{I}_i : | w(x)(\Phi(f)(x) - \Phi(p)(x))| \geqslant \| \Phi(f) - \Phi(p)\|/2 \\ \text{and } \operatorname{sgn}(f(x) - p(x)) \neq \operatorname{sgn}^*(f(x_i) - p(x_i))\} \end{aligned}$$

and set $Z = \bigcup_{i=1}^{k} Z_i$. Note that $| w(x)(\Phi(f)(x) - \Phi(p)(x))| < \| \Phi(f) - \Phi(p)\|$ for all $x \in Z$, $p(x) \neq l(x)$ and $p(x) \neq u(x)$ for each $x \in Z$ by construction of the intervals $\{I_i\}$. Finally, let

$$U_i = \{x \in X \cap \bar{I}_i : | w(x)(\Phi(f)(x) - \Phi(p)(x))| \leqslant \| \Phi(f) - \Phi(p)\|/2\}$$

and set $U = \bigcup_{i=1}^{k} U_i$. Then by continuity, there exists a λ_5 such that $0 < \lambda_5 < \lambda_4$ for which $x \in Z \cup U$ and $0 < \lambda \leqslant \lambda_5$ imply

$$| w(x)(\Phi(f)(x) - \Phi(p + \lambda q)(x))| < \| \Phi(f) - \Phi(p)\|$$

and

$$l(x) \leqslant p(x) + \lambda q(x) \leqslant u(x).$$

Combining all these results, we have that for $0 < \lambda \leqslant \lambda_5$,

$$\| \Phi(f) - \Phi(p + \lambda q)\| < \| \Phi(f) - \Phi(p)\|$$

and $p + \lambda q \in M$ establishing the Theorem.

COROLLARY 3. *In the above Theorem the best starting approximation for $\Phi(f)$ is unique and does exist.*

Proof. Uniqueness is a consequence of the above proof. Existence follows from the fact that M is a compact subset of $C(X)$.

Before continuing to our next characterization Theorem we would like to point out that Theorem 3 is true for more general functions l and u (see [15, 16]).

For the last example we fix points $\{y_i\}_{i=1}^{p}$ in X where $y_1 < y_2 < \cdots < y_p$, real numbers $\{a_i\}_{i=1}^{p}$ and define K by

$$K = \{f \in C(X): f(y_i) = a_i, i = 1, 2,..., p\}.$$

Let (a_{ij}) $j = 0, 1,..., m_i$, $i = 1,..., p$ be a second set of real numbers where $a_{i0} = a_i$, $1 \leqslant i \leqslant p$, $m = \sum_{i=1}^{p} (m_i + 1) \leqslant n$ and $\gamma = \max(m_i + 1)$. Let V be an n-dimensional extended Chebyshev space of order γ of $C[a, b]$ (see [5] and [9]) and define M by

$$M = \{p \in V: p^{(j)}(x_i) = a_{ij}, j = 0, 1,..., m_i ; i = 1,..., p\}.$$

THEOREM 4. *Let Φ: $K \to C(X)$ be a continuous operator which is pointwise strictly monotone and pointwise fixed at $f \in K \sim M$. Then $p \in M$ is a best starting approximation for $\Phi(f)$ if and only if there exists $\{x_i\}_{i=1}^{n-m+1} \subset X \sim \{y_1,..., y_p\}$ for which $x_1 < \cdots < x_{n-m+1}$,*

$$| w(x_i)(\Phi(f)(x_i) - \Phi(p)(x_i))| = \| \Phi(f) - \Phi(p)\|$$

and

$$\operatorname{sgn}\{(f(x_i) - p(x_i))\, \pi(x_i)\} = (-1)^{i-1} \operatorname{sgn}\{(f(x_1) - p(x_1))\, \pi(x_1)\}$$

for $i = 1,..., n - m + 1$ where $\pi(t) = (t - y_1)^{m_1} \cdots (t - y_p)^{m_p}$ if $p \neq 0$ and $\pi(t) \equiv 1$ if $p = 0$.

The proof of this theorem is patterned after the proof of the corresponding theorem in [9] and the previous two theorems and is therefore omitted.

Remark. Each of the above three theorems are true if we replace V by $R_m^n[a, b]$, the standard class of rationals functions normalized in the usual manner. The only change required is to change the number of characterizing extreme points to the same number that is needed in the standard Chebyshev approximation theory. The usual approximation problem corresponding to Theorem 3 may be found in [8] and [16]. The theory developed in these papers was referred to as approximation with rationals having restricted ranges. Some of the theory corresponding to Theorem 4 for rational functions can be found in [3, 7, and 16]. In these papers it is noted that there need not exist a best rational approximation satisfying interpolatory constraints. More general families could also be used. An example of such families can be found in a paper of Meinardus and Schwedt [11].

In the next section we shall apply this theory to some specific cases, corresponding to Theorem 2. For some of these cases we shall find that the best starting approximation for $\Phi(f)$ is simply a multiple of the best relative approximation to f. Also, in some special cases we shall find that repeated applications of Φ (if well defined) does not change the best starting value. Thus we shall close this section with a discussion of sufficient conditions on Φ for which this behavior occurs.

DEFINITION 3. We shall say that Φ possesses property I at $f \in K$ provided for each $p \in K$ and $x, y \in X$, $p(x)/f(x) = p(y)/f(y)$ implies $\Phi(p)(x)/f(x) =$

$\Phi(p)(y)/f(y)$, and $p(y)/f(y) < p(x)/f(x) \leqslant 1$ or $p(y)/f(y) > p(x)/f(x) \geqslant 1$ implies $| 1 - \Phi(p)(x)/f(x)| < | 1 - \Phi(p)(y)/f(y)|$.

DEFINITION 4. We shall say that Φ is one sided at f provided either $\Phi(k) \geqslant \Phi(f)$ for all $k \in K$ or $\Phi(k) \leqslant \Phi(f)$ for all $k \in K$.

THEOREM 5. *Let Φ: $K \to C(X)$, $\Phi(f) = f$ for some $f \in K$ where Φ, K, and M are as in Theorem 2 (V may be either an n-dimensional Haar subspace or $R_m{}^n[a, b]$), $f > 0$ on X, $C(X)$ be normed by $\| h \| = \| h/f \|_\infty$ and Φ possess property I at f. Let $p \in V$ be the best relative approximation to f from V and suppose $\|(f - p)/f \|_\infty = \lambda > 0$. If $\delta p \in M$ for $\delta \in [1/(1 + \lambda), 1/(1 - \lambda)]$ then there exists $\delta_0 \in (1/(1 + \lambda), 1/(1 - \lambda))$ for which $\delta_0 p$ is the best starting approximation for f (with respect to Φ).*

Proof. Let us first note that we necessarily have $\lambda < 1$ since arbitrarily small positive functions exist in V. Now from the standard theory we know that there exist points $\{x_i\}_{i=1}^{n+1} \subset X$ (the number of points depending upon V and possibly p) for which

(i) $x_1 < x_2 < \cdots < x_{n+1}$,

(ii) $|(f(x_i) - p(x_i))/f(x_i)| = \|(f - p)/p \|_\infty$,

(iii) $\operatorname{sgn}(f(x_i) - p(x_i)) = (-1)^{i-1} \operatorname{sgn}(f(x_1) - p(x_1))$.

We shall assume without lost of generality that $p(x_1) > f(x_1)$. Then $p(x_1)/(1 + \lambda) = f(x_1)$ and $p(x_2)/(1 - \lambda) = f(x_2)$. Also, for $x \in X$ we have that $\gamma p(x_1)/f(x_1) \geqslant \gamma p(x)/f(x) \geqslant \gamma p(x_2)/f(x_2)$ provided $\gamma \in [1/(1 + \lambda), 1/(1 - \lambda)]$. Thus by property I,

$$| 1 - \Phi(\gamma p)(x)/f(x)| \leqslant \max(| 1 - \Phi(\gamma p)(x_1)/f(x_1)|, | 1 - \Phi(\gamma p)(x_2)/f(x_2)|)$$

for all $x \in X$. Also, since $\gamma p(x_{i+2})/f(x_{i+2}) = \gamma p(x_i)/f(x_i)$ for $i = 1, 2,..., n - 1$ and $\gamma \in (1/(1 + \lambda), 1/(1 - \lambda))$ we have that $| 1 - \Phi(\gamma p)(x_{i+2})/f(x_{i+2})| = | 1 - \Phi(\gamma p)(x_i)/f(x_i)|$ for $i = 1, 2,..., n - 1$ and $\gamma \in (1/(1 + \lambda), 1/(1 - \lambda))$. Now as γ decreases to $1/(1 + \lambda)$ we have that $| 1 - \Phi(\gamma p)(x_1)/f(x_1)|$ decreases to 0 and $| 1 - \Phi(\gamma p)(x_2)/f(x_2)|$ increases (from 0) by the pointwise strict monotonicity of Φ. Likewise, as γ increases to $1/(1 - \lambda)$ we have that $| 1 - \Phi(\gamma p)(x_1)/f(x_1)|$ is a strictly increasing (from 0) function of γ and $| 1 - \Phi(\gamma p)(x_2)/f(x_2)|$ is a strictly decreasing function of γ with limit 0. Since both these forms are continuous functions of γ, we must have that there exists $\gamma_0 \in (1/(1 + \lambda), 1/(1 - \lambda))$ for which

$$| 1 - \Phi(\gamma_0 p)(x_1)/f(x_1)| = | 1 - \Phi(\gamma_0 p)(x_2)/f(x_2)|.$$

Finally, observing that for $\gamma \in (1/(1+\lambda), 1/(1-\lambda))$ we have

$$\operatorname{sgn}(f(x_i) - \gamma p(x_i)) = \operatorname{sgn}(f(x_i) - p(x_i)),$$

we conclude that $\gamma_0 p$ is the best starting approximation for f by Theorem 2.

THEOREM 6. *If Φ of Theorem 5 satisfies Φ: $K \to K$ and is also a one-sided operator then Φ^m has all the same properties as Φ and the best starting approximation for $\Phi(f)$ is also the best starting approximation for $\Phi^m(f)$ for $m = 2, 3,\ldots$.*

Proof. The fact that Φ^m has all the properties of Φ follows from Corollary 1 and a simple inductive application of property I. Suppose that $\Phi(h) \geqslant f$ for all $h \in K$. Then, we have at the points $x_1 < x_2 < \cdots < x_{n+1} \subset X$ characterizing the best starting approximation p for $\Phi(f)$ that

(i) $|(\Phi(p)(x_i) - f(x_i))/f(x_i)| = \|(\Phi(p) - f)/f\|_\infty$, $i = 1,\ldots, n+1$,

(ii) $\operatorname{sgn}(f(x_i) - p(x_i)) = (-1)^{i-1} \operatorname{sgn}(f(x_1) - p(x_1))$, $i = 1,\ldots, n+1$.

But applying property I to these points m times we see that p is also the best starting value for $\Phi^m(f) = f$. The other case ($\Phi(h) \leqslant \Phi(f)$ for all $h \in K$) follows in the same manner.

4. APPLICATIONS

1. Let

$$S = \{f \in C^2(0, \infty): f > 0 \text{ on } [a, b],\ 0 < a < b,\\ f' \text{ and } f'' \text{ do not vanish on } (0, \infty) \text{ and range } f = (0, \infty)\}.$$

For a given $x \in [a, b]$, the value of $f \in S$ can be found using Newton's method. That is, fix $x \in [a, b]$ and let $y(x)$ be a real number, then the sequence

$$\begin{aligned} y_0(x) &= y(x) \\ y_n(x) &= y_{n-1}(x) - (f^{-1}(y_{n-1}(x)) - x)(f'[f^{-1}(y_{n-1}(x))]), \end{aligned} \tag{1}$$

$n = 1, 2,\ldots$, is the Newton iteration for finding the unique zero of the equation

$$f^{-1}(y) - x = 0$$

starting with an initial guess of $y(x)$. This sequence converges (quadratically) to $y = f(x)$ provided $y(x)$ is sufficiently close to $f(x)$. Since we wish to calculate $f(x)$ for all $x \in [a, b]$ on a high speed digital computer, we shall select a class of functions M defined on $[a, b]$ each of which is easily programmed into our computer and use one of these functions as the initial guess. That is, we wish to find a $p \in M$ such that

$$\|(y_{n,p}(x) - f(x))/f(x)\|_\infty \leqslant \|(y_{n,q}(x) - f(x))/f(x)\|_\infty$$

for all $q \in M$, where $y_{n,q}(x)$ denotes the n-th Newton iterate at x starting with $q(x)$. Numerically, we would do this for functions f for which f^{-1} is easily evaluated, such as $f(x) = x^{1/N}$, $N =$ positive integers. This problem was studied by Moursund and Taylor [13] and is a generalization of the subroutine used to calculate $x^{1/2}$ on a high speed digital computer. We shall show that the theory developed in this paper may also be applied to this problem. In addition, we shall show that the behavior exhibited in Theorem 6 holds for $f(x) = x^\alpha$, $\alpha > 0$ [17] and also, for $f(x) = e^x$ which is a new result.

To set these results in the framework of our theory we must consider two cases.

Case 1. Fix $f \in S$ and assume that either $f' > 0$ and $f'' < 0$ on $(0, \infty)$ or $f' < 0$ and $f'' < 0$ on $(0, \infty)$. Here we set

$$K = \{h \in C[a, b]: h > 0 \text{ on } [a, b]\}$$

and

$$N_f(h)(x) = h(x) - \{f^{-1}(h(x)) - x\}\{f'[f^{-1}(h(x))]\}$$

for each $h \in K$. It is easily seen that $N_f: K \to K$ is continuous, $N_f(f) = f$, N_f is pointwise monotone at f, pointwise fixed at f and one sided from above at f.

Case 2. Fix $f \in S$ where either $f' > 0$ and $f'' > 0$ or $f' < 0$ and $f'' > 0$ holds on $(0, \infty)$. In this case the choice for K is somewhat more difficult. The problem here is that N_f is a one-sided operator from below and large values of $h(x)$ may give negative values for $N_f(h)(x)$. Since we wish to study N_f^m, $m = 1, 2, \ldots$, and we have assumed that the range of f is $(0, \infty)$, we must restrict the values of the functions in K so that the image of each of these functions under N_f is again in K. Setting

$$N(x, y) = y - \{f^{-1}(y) - x\}\{f'[f^{-1}(y)]\},$$

and calculating $\partial N(x, y)/\partial y$, we see that for x fixed $N(x, y)$ is a strictly increasing function of y for $0 < y < f(x)$ and a strictly decreasing function of y for $y > f(x)$. Thus by the implicit function Theorem, the equation $N(x, y) = 0$ defines $y = y(x) > f(x)$ as a continuous function of x. Setting

$$K = \{h \in C[a, b]: 0 < h(x) < y(x) \text{ for } x \in [a, b]\}$$

we have that $N_f: K \to K$ is continuous, $N_f(f) = f$, N_f is pointwise monotone at f, pointwise fixed at f and one-sided from below at f. Thus, we can state the following analog of Theorem 2, for both of these cases [13].

THEOREM 7. *Let $f \in S$, K be as above and V be a Haar subspace of $C[a, b]$ or $R_m^n[a, b]$. Set $M = V \cap K$. Then $p_m \in M$ is the unique best starting approximation for f with respect to the m-th Newton iteration if and only if there exists $\{x_i\}_{i=1}^{n+1} \subset [a, b]$, $x_1 < x_2 < \cdots < x_{n+1}$ (the length of this sequence depending on V and possibly p_m) for which*

$$\text{(i)}\quad |(f(x_i) - N_f^m(p_m)(x_i))/f(x_i)| = \|(f - N_f^m(p_m))/f\|_\infty, \qquad i = 1,..., n+1$$

and

$$\text{(ii)}\quad \operatorname{sgn}(f(x_i) - p(x_i)) = (-1)^{i+1} \operatorname{sgn}(f(x_1) - p(x_1)),\ i = 1,..., n+1.$$

Observe that nothing is said about the existence of a best starting approximation for f in the above theorem. As noted earlier, this is a difficult problem that must be studied separately. The existence of a best starting approximation will depend upon the function f, the interval $[a, b]$ and the class of approximants V. For the special case that $f(x) = x^\alpha$, $\alpha \in (0, 1)$ or $f(x) = e^x$ we can show that a best starting approximation exists. For the function $f(x) = x^\alpha$, the existence of the best starting approximation follows from the observation that

$$N_f(h)(x) = \alpha[(\alpha - 1)\, h(x) + x/h(x)^{1/\alpha - 1}]$$

approaches $+\infty$ as $h(x)$ approaches either 0 or $+\infty$. For this function, we need make no special assumptions on $[a, b]$ and V, other than the requirement $a > 0$.

For the function $f(x) = e^x$, this question is more difficult. Here we note that the function $\hat{y}(x) = e^{1+x}$ satisfies $N(x, \hat{y}(x)) = 0$ for all $x \in [a, b]$ where $N(x, y) = y(1 + x - \ln y)$. Thus, we define

$$K = \{h \in C[a, b]: 0 < h(x) < e^{1+x} \text{ for all } x \in [a, b]\}.$$

Let V be a Haar subspace of $C[a, b]$ or $R_m^n[a, b]$ and let $\bar{p} \in V$ be the best relative approximation to f with relative error λ. If we require $\lambda < (e - 1)/(e + 1)$, which is a requirement on V and the interval $[a, b]$, then we will be in a position to attempt to apply Theorem 5 to this problem. However, in the next Theorem we shall show that this restriction on λ is not necessary. That is, we shall show that a best starting approximation always exists for this function without special assumptions on V and $[a, b]$. It seems reasonable to expect that existence can always be obtained simply by requiring V to be sufficiently close to f, for each $f \in S$.

Next, we would like to note that for the special case that $f(x) = x^\alpha$, $\alpha > 0$ or $f(x) = e^x$ the best starting approximation is independent of m (the number of iterations) and is a positive multiple of the best relative approximation to f. (This result is actually true for $f(x) = \beta x^\alpha$, $\beta > 0$, $\alpha > 0$

and $f(x) = \gamma e^{\beta x}$, $\gamma > 0$ and $\beta > 0$). For the case when $f(x) = x^\alpha$ this result was proved in [17]. For $f(x) = e^x$ it is new. For use in the next theorem, we define $p_\alpha \in V$ and $p \in V$ by

$$\|(x^\alpha - p_\alpha)/x^\alpha\|_\infty = \inf_{q \in M} \|(x^\alpha - q)/x^\alpha\|_\infty = \lambda_\alpha$$

and

$$\|(e^x - p)/e^x\|_\infty = \inf_{q \in M} \|(e^x - q)/e^x\|_\infty = \lambda.$$

Note p_α and p are the best relative approximations to x^α and e^x, respectively, with deviations λ_α and λ.

THEOREM 8. *For $m = 1, 2, \ldots$, the following is true*

(a) *The best starting approximation for m Newton iterations for the calculation of x^α is $\gamma_\alpha p_\alpha$ where*

$$\gamma_\alpha = \left[\frac{(1+\lambda)^{\beta-1} - (1-\lambda)^{\beta-1}}{2(\beta-1)\,\lambda(1-\lambda^2)^{\beta-1}}\right]^\alpha, \qquad \beta = \frac{1}{\alpha}.$$

(b) *The best starting approximation for m Newton iterations for the calculation of e^x is γp where*

$$\begin{aligned}\gamma &= \exp[(1/2\lambda)(2\lambda + (1-\lambda)\ln(1-\lambda) - (1+\lambda)\ln(1+\lambda))] \\ &= ((1-\lambda)/(1+\lambda))^{1/2\lambda}(1-\lambda^2)^{-1/2}.\end{aligned}$$

Remark. That is, suppose one wishes to calculate the value of x^α (some $\alpha > 0$ or e^x) on an interval $[a, b]$ using the following scheme. Program a function $q \in M$ into the machine. Calculate m Newton iterations on the function $f(y) = y^{1/\alpha} - x$ using the sequence defined by (1), starting with the value $q(x)$. Use the m-th iterate as an approximation to x^α. Then, if one starts with $\gamma_\alpha p_\alpha$, the m-th iterate is a better relative approximation to x^α than is the m-th iterate starting with any other function $q \in M$. This is how Cody [2] suggested that one evaluate $x^{1/2}$ in double precision on a CDC 3600. (that his suggested starting approximation is actually ours was pointed out in [14]).

Proof. Since the first result was proved in [17], we shall only prove the second statement. Actually, all that one must do to get this result is show that $N \equiv N_{e^x}$ possesses property I and apply Theorem 2. Property I follows from observing that

$$N(p)(x) = p(x)[1 - \ln(p(x)/e^x)]$$

so that $p(y)/e^y = p(z)/e^z$ clearly implies that $N(p)(y)/e^y = N(p)(z)/e^z$. Setting

$$\varphi(t) = t[1 - \ln t]$$

and noting that $\varphi(1)$ is the only local maximum for $\varphi(t)$ when $t > 0$, we see that the second condition of property I is also satisfied. Next, let us show that the best relative approximation p to e^x from V belongs to

$$K = \{h \in C[a, b]: 0 < h(x) < e^{1+x} \text{ on } [a, b]\}.$$

This follows from the fact that $(1 - \lambda)\, e^x \leqslant p(x) \leqslant (1 + \lambda)\, e^x$ where λ the relative error satisfies $0 < \lambda < 1$. Using property I, as in the proof of Theorem 5, we find that γp has an error curve of the type described in Theorem 2 where

$$\gamma = ((1 - \lambda)/(1 + \lambda))^{1/2\lambda}(1 - \lambda^2)^{-1/2}.$$

Since $0 < \gamma < 1$, we have that $\gamma p \in K$. Thus, by Theorem 2, γp is the best starting approximation for e^x with respect to N. Since $\gamma p \in K$ we have that $N(\gamma p) \in K$. From this and by property I and the convergence properties of the Newton iteration for e^x, we immediately get that γp is the best starting approximation for e^x with respect to N^m, $m = 1, 2,\ldots$, by Theorem 2.

2. The final application that we shall consider concerns some recent work by Merz [10]. Merz has shown that the iterative scheme

$$\varphi_k(y) = x^{1/2}\, \frac{(y + x^{1/2})^k + (y - x^{1/2})^k}{(y + x^{1/2})^k - (y - x^{1/2})^k}$$

for a fixed integer $k \geqslant 2$ defines a sequence converging to $x^{1/2}$ starting with any $y > 0$ (i.e., $y_0 = y$, $y_n = \varphi_k(y_{n-1})$ is such that $y_n \to x^{1/2}$). Furthermore, the order of this convergence is k. That is,

$$\lim_{n\to\infty}[(y_n - x^{1/2})/(y_{n-1} - x^{1/2})^{k+1}] = 0.$$

For $k = 2$, $\varphi_2(y) = (1/2)(y + x/y)$ which is the standard Newton iteration. Also, the formulas for larger k are quite nice. For example,

$$\varphi_7(y) = (y^7 + 21xy^5 + 35x^2y^3 + 7x^3y)/(7y^6 + 35xy^4 + 21x^2y^2 + x^3).$$

These schemes possess the interesting property that

$$\varphi_k(\varphi_m(y)) = \varphi_{k\cdot m}(y)$$

so that one has the choice of repeated iterations with a lower order formula or less iterations a higher order formula to obtain the same final iterate.

We wish to phrase this method of calculating $x^{1/2}$ into the terminology of our paper and, thereby, obtain the best starting approximations for these schemes. Thus, let us define Φ_k: $K \to K$ (K the positive continuous functions) by

$$\Phi_k(h)(x) = x^{1/2}\, \frac{(h(x) + x^{1/2})^k + (h(x) - x^{1/2})^k}{(h(x) + x^{1/2})^k - (h(x) - x^{1/2})^k}$$

for each $h \in K$. It is easily seen that Φ_k is continuous and $\Phi_k(x^{1/2}) = x^{1/2}$. By considering the function

$$\varphi_k(t) = ((1+t)^k + (1-t)^k)/((1+t)^k - (1-t)^k)$$

for $t > 0$ and computing its derivative, one easily sees that for k even, Φ_k is pointwise strictly monotone at $x^{1/2}$, one sided at $x^{1/2}$ (from above) and possesses property I at $x^{1/2}$. Likewise, for k odd, Φ_k is pointwise strictly monotone at $x^{1/2}$ and possesses property I at $x^{1/2}$. Thus, by Lemma 3 and Theorem 5 we know that for each k, the best starting approximation for $x^{1/2}$ on $[a, b]$ with respect to Φ_k from M (as described in Theorem 8) is a positive multiple of the best relative approximation to $x^{1/2}$ on $[a, b]$ from M. Furthermore, if k is even then Φ_{k^m} for $m = 1, 2, \ldots$, has the same starting value for all m by Theorem 6 as $\Phi_{k^m}(h) = \Phi_k{}^m(h) = \Phi_k(\Phi_k^{m-1}(h))$.

Now for k even, we have that

$$\Phi_k(q)(x)/x^{1/2} \geqslant 1$$

for all $q \in M$ and $x \in [a, b]$. Letting p be the best relative approximation to $x^{1/2}$ on $[a, b]$ with deviation λ,

$$\|(x^{1/2} - p(x))/x^{1/2}\|_\infty = \inf_{q \in M} \|(x^{1/2} - q(x))/x^{1/2}\|_\infty = \lambda.$$

we know by Theorem 5 that exists a unique $\gamma \in (1/(1+\lambda), 1/(1-\lambda))$ such that γp is the best starting approximation for $x^{1/2}$ on $[a, b]$ with respect to Φ_{k^m} for all $m = 1, 2, \ldots$, and γ can be found by solving

$$(\Phi_k(\gamma p)(x_1)/x_1^{1/2}) - 1 = (\Phi_k(\gamma p)(x_2)/x_2^{1/2}) - 1$$

where x_1 and x_2 are points at which $p(x_1) = (1+\lambda)\, x_1^{1/2}$ and $p(x_2) = (1-\lambda)\, x^{1/2}$. This equation leads to the simpler equation

$$(\gamma(1+\lambda) - 1)^k(\gamma(1-\lambda) + 1)^k = (\gamma(1+\lambda) + 1)^k(\gamma(1-\lambda) - 1)^k.$$

Using the fact that k is even and the solution we desire belongs to $(1/(1+\lambda), 1/(1-\lambda))$, we obtain

$$\gamma = (1/(1-\lambda^2))^{1/2}.$$

Thus *every* even ordered scheme has the same best starting approximation namely $(1/(1-\lambda^2))^{1/2}\, p$.

For k odd, we must start with the equation

$$(\Phi_k(\gamma p)(x_1)/x_1^{1/2}) - 1 = 1 - \Phi_k(\gamma p)(x_2)/x_2^{1/2}$$

This reduces to

$$((1 - \lambda\gamma + \gamma)/(1 + \lambda\gamma - \gamma))^k + ((1 + \lambda\gamma + \gamma)/(1 - \lambda\gamma - \gamma))^k = 2$$

and the unique solution γ in the interval $(1/(1 + \lambda), 1/(1 - \lambda))$ has to be found here (apparently) by numerical methods on a computer.

Combining the above results, we have the following

THEOREM 9. *Let p be the best relative approximation to $x^{1/2}$ on $[a, b]$, $0 < a < b$ from M and set $\lambda = \|(p(x) - x^{1/2})/x^{1/2}\|_\infty$. Let $q_m(x)$ be the approximation to $x^{1/2}$ defined as follows*:

$$q_0(x) = q(x), \qquad q \in M,$$
$$q_n(x) = \Phi_k(q_{n-1})(x), \qquad n = 1, 2,..., m,$$

where k is a fixed positive integer and Φ_k is defined above. Then $q_m(x)$ is the unique best relative approximation to $x^{1/2}$ on $[a, b]$, $0 < a < b$ from $\Phi_k{}^m(M)$ if and only if $q_m(x) = \gamma_m p(x)$ where

$$\gamma_m = (1/(1 - \lambda^2))^{1/2}$$

if k is even and γ_m is the only solution of

$$((1 - \lambda\gamma_m + \gamma_m)/(1 + \lambda\gamma_m - \gamma_m))^{k^m} + ((1 + \lambda\gamma_m + \gamma_m)/(1 - \lambda\gamma_m - \gamma_m))^{k^m} = 2$$

in the interval $(1/(1 + \lambda), 1/(1 - \lambda))$ *for k odd.*

Remark. Existence and uniqueness of $\gamma_m p$ follows from Theorem 5. The fact that Φ_k is one-sided for k even implies γ_m is independent of m and actually solving for γ_m shows that γ_m is also independent of k.

REFERENCES

1. E. W. CHENEY, "Introduction to Approximation Theory," McGraw-Hill, New York, 1966.
2. W. J. CODY, Double-precision square root for the CDC-3600, *Comm. ACM* **7** (1964), 715–718.
3. M. CLAUDE GILORMINI, Sur un countre-exemple de M. Henry L. Loeb, *C. R. Acad. Sci., Paris, Ser. A* **266** (1968), 239–241.
4. A. HOLZWARTH, Ein Verfahren zur Bestimmung bester Tschebyscheff, Approximationen der Quadratwurzelfunktion, *Computing*, **4** (1969), 168–177.
5. S. J. KARLIN AND W. J. STUDDEN, "Tchebycheff Systems: With Applications in Analysis and Statistics," Interscience, New York, 1966.
6. R. F. KING AND D. L. PHILLIPS, The logarithmic error and Newton's method for the square root, *Comm. ACM* **12** (1969), 87–88.

7. H. L. LOEB, Un contre-example a un résultat de M. Claude Gilnormini, *C. R. Acad. Sc., Paris, Ser. A* **266** (1968), 237–238.
8. H. L. LOEB, D. G. MOURSUND AND G. D. TAYLOR, Uniform rational weighted approximations having restricted ranges, *J. Approximation Theory* **1** (1968), 401–411.
9. H. L. LOEB, D. G. MOURSUND, L. L. SCHUMAKER, AND G. D. TAYLOR, Uniform generalized weight function polynomial approximation with interpolation, *SIAM J. Numer. Anal.* **6** (1969), 284–293.
10. A. MERZ, Thesis, Der Universität Erlangen, 1968.
11. G. MEINARDUS AND D. SCHWEDT, Nicht-Lineare Approximationen, *Arch. Rational Mech. Anal.* **17** (1964), 297–326.
12. D. G. MOURSUND, Optimal starting values for the Newton-Raphson calculation of $x^{1/2}$. *Comm. ACM* **10** (1967), 430–432.
13. D. G. MOURSUND AND G. D. TAYLOR, Optimal starting values for the Newton-Raphson calculation of inverses of certain functions, *SIAM J. Numer. Anal.* **5** (1968), 138–150.
14. P. H. STERBENZ AND C. T. FIKE, Optimal starting approximations for Newton's method, *Math. Comp.* **23** (1969), 313–318.
15. G. D. TAYLOR, Approximation by functions having restricted ranges III, *J. Math. Anal. Appl.* **27** (1969), 241–248.
16. G. D. TAYLOR, Approximation by functions having restricted ranges: equality case, *Numer. Math.* **14** (1969), 71–78.
17. G. D. TAYLOR, Optimal starting approximations for Newton's method, *J. Approximation Theory* **3** (1970), 156–163.